A note on necessary conditions for phase memory

1979 J. Phys. A: Math. Gen. 12 2343

(http://iopscience.iop.org/0305-4470/12/12/013)

J. Phys. A: Math. Gen., Vol. 12, No. 12, 1979. Printed in Great Britain

A note on necessary conditions for phase memory

J Heading

Department of Applied Mathematics, The University College of Wales, Aberystwyth, Dyfed, UK

Received 23 January 1978, in final form 6 September 1978

Abstract. Misconceptions still exist regarding the nature of the approximation behind the so-called WKBJ solutions to wave propagation problems, particularly when the second-order differential equation is not in normal form, and when the order of the equation is higher than two. Here we examine (i) necessary conditions via the differential equation for the validity of the additional phase memory terms, and their possible transformation into local factors, showing how these terms arise and when they are meaningful, and (ii) sufficient conditions briefly via the corresponding integral equations. Investigation (i) is based on two sets of inequalities, the first between the moduli of the approximate solutions of the corresponding first-order equations, and the second between the moduli of the exponentials of the integrals of the effective refractive indices.

1. The concept of phase memory

The ultimate goal of the theory of the WKBJ or phase-integral solutions of a (second-order) differential equation in the complex plane has been attained in the recent paper by Olver (1978). Many readers intent upon using these solutions in physical problems (such as the propagation of radio waves in an anisotropic ionosphere) may be daunted when faced with such an investigation on account of the complicated analysis necessary to justify the extension of the theory into the complex plane when several transition points are involved. A more popular account of how the WKBJ solutions may be extended around transition points in the complex plane has been given by Heading (1977).

On the other hand, at the mathematical level used by many investigators in their application of these solutions, clarity is still needed regarding the forms of the approximate solutions in domains not containing transition points. This may appear surprising but the reason may be seen from the following assessment.

For a second-order differential equation in normal form

$$d^2w/dz^2 + k^2q^2(z)w = 0,$$

the WKBJ solutions for large k are well known to be

$$w = q^{-1/2}\exp\left(\pm ik\int q\, dz\right),$$

where $k\int q\, dz$ represents the phase memory, and $q^{-1/2}$ a local factor. An error analysis based on an equivalent integral equation reveals the nature of the approximation; terms of the form $O(1/k)$ are associated with the WKBJ solutions and are uniformly valid throughout restricted domains in the complex z plane; see Jeffreys (1962), Heading (1962).

To omit the factor $q^{-1/2}$ and to use only $w = \exp(\pm ik \int q \, dz)$ is not correct, since the error analysis based on the integral equation can only be applied to the complete approximate solution.

Often, the one second-order equation is written as two first-order equations and matrix techniques are used to transform them suitably so as to extract the approximate WKBJ solutions; see Heading (1962, 1975). This method clearly shows the origin of the phase memory and the local factor and by the very form of the first-order equations (arising from a second-order equation in normal form), no other phase-memory term is possible. But the local factors arise from the neglect of other terms in the first-order equations and this process is usually taken for granted, regardless of whether the neglected terms are genuinely small compared with terms giving rise to the local factors, and this can lead to serious conceptual and mathematical mistakes. In their comprehensive survey of the subject Berry and Mount (1972) are not concerned with the treatment of phase memory, because this only becomes important when equations other than normal second-order equations are investigated. Budden's survey (1975) mentions the subject only in a closing brief paragraph.

Smith (1975) deals with the subject from the point of view of anisotropic ionospheric radio propagation. The appropriate WKBJ solutions of the four first-order equations are found to be

$$w_i = \exp\left(-ik \int q_i \, dz + k \int \Gamma_{ii} \, dz\right)$$

by an argument based on the smallness of the remaining terms Γ_{ij} $(i \neq j)$, a criterion which, as will be pointed out generally, cannot be simultaneously correct for all four solutions. Generally speaking, the integral $\int \Gamma_{ii} \, dz$ cannot be manipulated so as to yield a local term, though part of it can; the remainder must be retained as an additional phase-memory. Conditions for the integrability of $\int \Gamma_{ii} \, dz$ are examined in the paper.

Budden and Smith (1976) continue the investigation of this additional phase memory as applied to various types of wave propagation in geophysics and atmospheric physics and find integrable cases when only local factors are produced. For second-order equations, they consider waves in an isotropic ionosphere, in an isotropic magnetic dielectric, in an optically active dielectric and atmospheric gravity waves. For higher-order equations, they consider electromagnetic waves in a cold anisotropic plasma, electro-acoustic waves in isotropic warm plasma, waves in an optically active medium, seismic waves with and without the effect of a coriolis force and magneto-hydrodynamic waves.

In the present paper, we are concerned with the origin of this term Γ_{ii} in the approximate solutions, regardless of whether it can be wholly or partially integrated so as to introduce a local factor or to remain as an additional phase-memory term. Approximations are possible by taking a domain throughout which there are to be satisfied certain inequalities between terms in the differential equations. Approximate solutions are then derived when these small terms are neglected. Some investigators never pursue the analysis beyond this point. But further investigation is essential. If this further analysis is to be based solely on the differential equations, the approximate solutions must be used to examine the consistency of the assumed inequalities, for it may turn out that they cannot be valid when the approximate solutions are used. This consistency depends in turn on other inequalities (or equalities) between the moduli of

$\exp(\int q_i\, dz)$. This interlocking of inequalities provides a basis for deriving the WKBJ solutions simply from the first-order equations and for deciding whether such solutions may or may not be taken in linear combinations. It is the author's opinion that much spurious mathematics (sometimes nevertheless yielding correct results) could have been avoided by investigators realising the importance of this kind of approach which yields necessary but not sufficient conditions for the validity of the WKBJ solutions. Sufficiency is dealt with by means of equivalent integral equations in which the terms previously neglected are retained in the integrals involved; these equations are solved by an iterative process using successive substitution, and estimates of the magnitudes of successive terms are obtained, thereby yielding uniformly convergent series and establishing the $O(1/k)$ error estimates.

Our intention in this paper is to concentrate on the necessary conditions via the differential equations and to introduce a particular step in the analysis which distinguishes a term that may be neglected from one that may not be, although both may be of the same order of magnitude. At the same time the integral equations are not overlooked since these provide the ultimate sufficiency behind the approximation process.

2. The first-order equations

We commence our investigation with a general second-order linear differential equation not in normal form,

$$d^2w/dz^2+2ku(z)\,dw/dz+k^2v(z)w=0,$$

where k is a large real parameter and $u(z)$ and $v(z)$ suitable functions of z; they may also be functions of k with inverse powers of k allowed. In matrix form we write

$$\begin{pmatrix} w \\ w' \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -k^2v & -2ku \end{pmatrix}\begin{pmatrix} w \\ w' \end{pmatrix},$$

or

$$\boldsymbol{w}' = \boldsymbol{T}\boldsymbol{w},$$

say, where a prime denotes differentiation with respect to z.

We introduce the change of variable $\boldsymbol{w} = \boldsymbol{R}\boldsymbol{f}$, where

$$\boldsymbol{R} = \begin{pmatrix} 1 & 1 \\ q_1 & q_2 \end{pmatrix},$$

q_1 and q_2 being the characteristic roots of $\boldsymbol{T}$. The characteristic equation is

$$q^2+2kuq+k^2v=0,$$

with roots

$$q=k[-u\pm\sqrt{(u^2-v)}],$$

which we shall write as $k(-u\pm D^{1/2})$. Then

$$\boldsymbol{f}' = \boldsymbol{Q}\boldsymbol{f} - \boldsymbol{R}^{-1}\boldsymbol{R}'\boldsymbol{f}, \tag{1}$$

where

$$\boldsymbol{Q}=\begin{pmatrix} q_1 & 0 \\ 0 & q_2 \end{pmatrix}$$

and

$$\boldsymbol{R}^{-1}\boldsymbol{R}'=\begin{pmatrix} q_1' & q_2' \\ -q_1' & -q_2' \end{pmatrix}\Big/(q_1-q_2).$$

We shall call the elements of $\boldsymbol{\Gamma}\equiv-\boldsymbol{R}^{-1}\boldsymbol{R}'$ *coupling coefficients* and those of $\boldsymbol{\Gamma f}$ *coupling terms.*

The usual method of procedure is to solve the simultaneous equations (1) by neglecting some of the coupling terms, a process that should depend on the relative order of magnitude of the terms involved. The introduction of the concept of phase memory needs clarification when approximate solutions are sought, since claims are sometimes made that cannot be justified, and yet that seem to yield the expected results!

Written explicitly, the simultaneous equations (1) are

$$f_1'=q_1f_1+\Gamma_{11}f_1+\Gamma_{12}f_2, \qquad f_2'=q_2f_2+\Gamma_{21}f_1+\Gamma_{22}f_2. \tag{2}$$

Corresponding integral equations may be simply developed based on the integrating-factor method for first-order linear equations. One pair is

$$\begin{aligned} f_1&=\exp\left(\int_{z_0}^{z} q_1\,\mathrm{d}\zeta\right)\left[A_1+\int_{z_0}^{z}\exp\left(-\int_{z_0}^{\zeta} q_1\,\mathrm{d}\zeta'\right)(\Gamma_{11}f_1+\Gamma_{12}f_2)\,\mathrm{d}\zeta\right],\\ f_2&=\exp\left(\int_{z_0}^{z} q_2\,\mathrm{d}\zeta\right)\left[A_2+\int_{z_0}^{z}\exp\left(-\int_{z_0}^{\zeta} q_2\,\mathrm{d}\zeta'\right)(\Gamma_{21}f_1+\Gamma_{22}f_2)\,\mathrm{d}\zeta\right], \end{aligned} \tag{3}$$

while another pair is

$$\begin{aligned} f_1&=\exp\left(\int_{z_0}^{z} (q_1+\Gamma_{11})\,\mathrm{d}\zeta\right)\left[A_1+\int_{z_0}^{z}\exp\left(-\int_{z_0}^{\zeta} (q_1+\Gamma_{11})\,\mathrm{d}\zeta'\right)\Gamma_{12}f_2\,\mathrm{d}\zeta\right],\\ f_2&=\exp\left(\int_{z_0}^{z} (q_2+\Gamma_{22})\,\mathrm{d}\zeta\right)\left[A_2+\int_{z_0}^{z}\exp\left(-\int_{z_0}^{\zeta} (q_2+\Gamma_{22})\,\mathrm{d}\zeta'\right)\Gamma_{21}f_1\,\mathrm{d}\zeta\right], \end{aligned} \tag{4}$$

the constants being determined by the boundary conditions of the problem.

3. The first defective approximation

The most naive approach has been to neglect all coupling terms $\boldsymbol{\Gamma f}$, on the grounds that the principal terms $\boldsymbol{Qf}$ on the right-hand side of (1), each with a factor k, are large compared with the coupling terms. This would give

$$f_i=A_i\exp\left(\int q_i\,\mathrm{d}z\right), \tag{5}$$

suggesting that both solutions are independent (except at points where $q_1=q_2$, namely when Γ is singular, at the *coupling, reflection* or *transition* points). The reason why this argument is so defective is because no uniform error bounds throughout particular specified domains can be arranged to be associated with such solutions (2). It is equivalent to suggesting that good approximate asymptotic expressions for the Airy integral and its companion function can have the forms $\exp(\pm\frac{2}{3}z^{3/2})$ rather than $z^{-1/4}\exp(\pm\frac{2}{3}z^{3/2})$.

The iterative method of solution is commenced by substituting (5) into the first equation (4), yielding a correction term containing two integrals. The first is obviously $O(1)$, while the second may be manipulated by integration by parts to yield an estimate $O(1/k)$ uniformly with respect to z. Overall therefore we see the futility even at this stage of the iterative process of suggesting that (5) is in any sense an approximate solution of equation (1), although the neglect of all the coupling terms may appear attractive on the surface.

4. The second defective approximation

This is usually found by retaining the two diagonal coupling terms, but neglecting the non-diagonal coupling terms, giving

$$f_1 = A_1 \exp\left[\int \left(q_1 - \frac{q_1'}{q_1 - q_2}\right) dz\right], \qquad f_2 = A_2 \exp\left[\int \left(q_2 - \frac{q'}{q_2 - q_1}\right) dz\right], \tag{6}$$

again suggesting that these are always independent. Using $q_{1,2} = k(-u \pm D^{1/2})$ we have

$$f_1 = A_1 D^{-1/4} \exp\left[\int \left(k(-u + D^{1/2}) + \frac{u'}{2D^{1/2}}\right) dz\right],$$

$$f_2 = A_2 D^{-1/4} \exp\left[\int \left(k(-u - D^{1/2}) - \frac{u'}{2D^{1/2}}\right) dz\right].$$

In these approximations, $D^{-1/4}$ is known as the local factor, while $\int [k(-u \pm D^{1/2}) \pm u'/2D^{1/2}]\, dz$ are known as phase-memory terms, showing that the change of phase is cumulative as a wave-like solution propagates through the medium.

It must be admitted that such a method for finding these approximations is quite untenable, whether for second- or for fourth-order equations, even though sometimes they are perfectly in order. If subsequent numerical investigations show that further results derived from such approximations are in good agreement with exact solutions, this still does not justify the method of approximation since no examination is made of the errors involved. Even without this examination, the argument is fallacious! When the approximations are introduced properly into the differential equations, quite a different understanding is gained of the origin of these local and phase-memory terms, and also of the independence or otherwise of the proposed solutions (6).

In the following section, we shall use an argument based on differential equations only to derive necessary conditions for the validity or otherwise of the proposed solutions (6). First, however, by means of integral equations (4), we examine the solutions along an anti-Stokes line, defined as follows.

With a suitable lower limit for the phase-memory terms, a line in the complex plane along which

$$\mathrm{Re} \int q_1\, dz = \mathrm{Re} \int q_2\, dz$$

is known as an anti-Stokes line. Along such a line, $\exp(\int q_1\, dz)$ and $\exp(\int q_2\, dz)$ are of equal modulus. At points not on such lines, when k is large one of these exponentials is large in magnitude and one is small.

When possible, consider integration through a domain in which the moduli of

$$\Gamma_{11}f_1,\ \Gamma_{12}f_2,\ \Gamma_{21}f_1,\ \Gamma_{22}f_2 \tag{7}$$

are all of the same order of magnitude with respect to k. Under these circumstances in the differential equations none of the coupling terms can be neglected in comparison with any other, unlike what is often asserted. Although no distinction can be drawn between diagonal and non-diagonal coupling terms, yet in the integral equations (4) the non-diagonal coupling terms are relegated to the second integrals.

The standard iterative process is commenced by substituting the two exponentials (6) into (4). The second terms are integrated by parts, after which it is deduced directly that the terms are of the form $O(1/k)$ uniformly with respect to z as $k \to \infty$. The process is repeated ultimately yielding convergent Liouville–Neumann expansions (see Jeffreys 1962). This gives the impression that the non-diagonal coupling terms in the differential equations have been neglected in comparison with the diagonal coupling terms of the same order of magnitude. This illusory success in this case in no way assures similar success elsewhere than along an anti-Stokes line.

5. Valid approximations for necessary conditions

From the point of view of the differential equations (2), any neglect of a coupling term is dictated by the order of magnitude of the coupling term when compared with the others and also with the principal terms $q_i f_i$, and such order of magnitude differences are maintained uniformly through some domain of the complex z plane. When any assumption is made that permits the neglect of a particular coupling terms then the ensuing approximate solutions must be used to test the consistency of the assumption either analytically or numerically. In this connection it is wrong, as is often done, to consider merely the magnitude of the coupling coefficients $\boldsymbol{\Gamma}$; the complete coupling terms $\boldsymbol{\Gamma f}$ must be examined.

A legitimate procedure for the neglect of a coupling term is first to introduce into equations (2) a change of dependent variable

$$\boldsymbol{f} = \exp\left(\int_{z_0}^{z} \Gamma_{11}\, d\zeta\right)\boldsymbol{g},$$

say, giving

$$g_1' = q_1 g_1 + \Gamma_{12} g_2, \qquad g_2' = q_2 g_2 + \Gamma_{21} g_1 + (\Gamma_{22} - \Gamma_{11}) g_2. \tag{8}$$

Both the diagonal coupling terms cannot be removed by this means, unless $\Gamma_{11} = \Gamma_{22}$ when the original differential equation is normal in form.

We need not now suppose any inequality between the coupling terms. Since k is a factor of q_1 but not of Γ_{12}, it is now legitimate to consider the proposed inequality throughout a domain

$$|q_1 g_1| \gg |\Gamma_{12} g_2|, \tag{9}$$

enabling us to write approximately $g_1' = q_1 g_1$, with solution

$$g_1 = A_1 \exp\left(\int_{z_0}^{z} q_1\, d\zeta\right)$$

and

$$f_1 = A_1 \exp\left(\int_{z_0}^{z} \Gamma_{11}\, d\zeta\right) \exp\left(\int_{z_0}^{z} q_1\, d\zeta\right),$$

identical with (6). We now substitute this into (8) without any further approximation being introduced, yielding a simple first-order linear equation for g_2 with solution

$$g_2 = \exp\left(\int_{z_0}^{z} (q_2 + \Gamma_{22} - \Gamma_{11})\, d\zeta\right) \int_{z_0}^{z} \exp\left(\int_{z_0}^{\zeta} (-q_2 - \Gamma_{22} + \Gamma_{11})\, d\zeta'\right) \Gamma_{21} g_1\, d\zeta$$
$$+ A_2 \exp\left(\int_{z_0}^{z} (q_2 + \Gamma_{22} - \Gamma_{11})\, d\zeta\right). \tag{10}$$

If the values of q_1 and q_2 satisfy the condition for integration along an anti-Stokes line, and if in g_2 the first term is of the form $O(1/k)$, then

$$g_2 = A_2 \exp\left(\int_{z_0}^{z} (q_2 + \Gamma_{22} - \Gamma_{11})\, d\zeta\right)$$

to this order of approximation. The whole process is finally seen to be self-consistent, since along an anti-Stokes line the four quantities (7) are all of the same order of magnitude with respect to k, forming with (9) necessary conditions for the validity of the approximation. In other words, these results for g_1 and g_2 are those that would be obtained by the neglect of the non-diagonal coupling terms in (2), but a legitimate use of inequalities has been employed to demonstrate the necessary conditions. Certainly the diagonal coupling terms must be used, producing both local terms and an addition to the phase-memory terms beyond $\int q_1\, dz$ and $\int q_2\, dz$. Sufficiency must be considered by means of the integral equations as dealt with in paragraph 4.

To this order of accuracy,

$$\begin{pmatrix} w \\ w' \end{pmatrix} = \boldsymbol{w} = \boldsymbol{Rf}$$
$$= \begin{pmatrix} A_1 \exp\left(\int_{z_0}^{z} (q_1 + \Gamma_{11})\, d\zeta\right) + A_2 \exp\left(\int_{z_0}^{z} (q_2 + \Gamma_{22})\, d\zeta\right) \\ A_1 q_1 \exp\left(\int_{z_0}^{z} (q_1 + \Gamma_{11})\, d\zeta\right) + A_2 q_2 \exp\left(\int_{z_0}^{z} (q_2 + \Gamma_{22})\, d\zeta\right) \end{pmatrix}$$

showing to this order of accuracy (i) that a linear combination of the two approximate solutions is permitted along an anti-Stokes line, and (ii) that when $\boldsymbol{w}$ is differentiated only the principal terms and not the diagonal coupling coefficients in the integrals must be differentiated.

Consider now how the differential equations can be handled so as to yield necessary conditions when a domain is selected not involving an anti-Stokes line in which $|f_1| \gg |f_2|$. The coupling coefficients are still of the same order of magnitude but the coupling terms will satisfy the inequalities

$$|\Gamma_{11} f_1| \gg |\Gamma_{12} f_2|, \qquad |\Gamma_{21} f_1| \gg |\Gamma_{22} f_2|, \tag{11}$$

so we may write equations (2) approximately as

$$f_1' = q_1 f_1 + \Gamma_{11} f_1, \qquad f_2' = q_2 f_2 + \Gamma_{21} f_1.$$

Note that both non-diagonal coupling terms cannot be neglected at the same time, unlike the previous case when not even one coupling term could be neglected initially until a transformation was made.

The solution for f_1 has the same apparent form as previously, and f_2 will be

$$f_2 = \exp\left(\int_{z_0}^{z} q_2\,\mathrm{d}\zeta\right)\int_{z_0}^{z}\exp\left(-\int_{z_0}^{\zeta} q_2\,\mathrm{d}\zeta'\right)\Gamma_{21}f_1\,\mathrm{d}\zeta + A_2\exp\left(\int_{z_0}^{z} q_2\,\mathrm{d}\zeta\right).$$

If as usual the first term in f_2 is of the form $f_1\mathrm{O}(1/k)$ uniformly throughout the domain, two cases arise.

(i) If

$$\left|\exp\left(\int_{z_0}^{z} q_1\,\mathrm{d}\zeta\right)\right| \gg \left|\exp\left(\int_{z_0}^{z} q_2\,\mathrm{d}\zeta\right)\right|$$

for large k, then

$$f_2 = f_1\mathrm{O}(1/k),$$

and inequalities (11) are satisfied; this is the dominant solution throughout the domain.

(ii) If

$$\left|\exp\left(\int_{z_0}^{z} q_1\,\mathrm{d}\zeta\right)\right| \ll \left|\exp\left(\int_{z_0}^{z} q_2\,\mathrm{d}\zeta\right)\right|$$

for large k, then a subdominant solution is produced provided $A_2 = 0$, yielding again

$$f_2 = f_1\mathrm{O}(1/k),$$

with inequalities (11) again satisfied.

It will be noticed that $\exp(\int q_2\,\mathrm{d}z)$ enters neither solution, except on an anti-Stokes line and when, of course, the roles of q_1 and q_2 are reversed in some other domain. Only an argument of this kind will indicate conclusively necessary conditions when and why some coupling terms can be neglected. Linear combinations of the two exponential forms are not allowed. The necessary inequalities are supplemented by considerations based on the integral equations in which the path of integration must be chosen in keeping with the texts, for example, by Jeffreys (1962) and Heading (1962), so as to preserve uniformity in the errors associated with the first approximations.

6. Fourth-order equations

Let the four first-order coupled equations be written in the form

$$f_i' = q_i f_i + \sum_{j=1}^{4} \Gamma_{ij} f_j,$$

where k occurs as a factor in the q_i but not in the coupling coefficients Γ_{ij}. Whether any of the coupling terms can be neglected throughout a specified domain depends on the relative order of magnitude of the sixteen terms $\Gamma_{ij}f_j$, and generally more possibilities exist than in the second-order case.

(i) If $|f_1| \gg |f_2|, |f_3|, |f_4|$ throughout the domain, we can approximate the equations as

$$f_1' = q_i f_i + \Gamma_{i1} f_1,$$

whose solutions involve constants A_1, A_2, A_3, A_4 respectively. These solutions are usually

$$\begin{aligned} &f_1 = A_1 \exp\left(\int (q_1 + \Gamma_{11})\,\mathrm{d}z\right), \\ &f_j = f_1 \mathrm{O}(1/k), \qquad j = 2, 3, 4, \end{aligned} \tag{12}$$

a dominant solution (or possessing a fourth level of dominancy) when

$$\left|\exp\left(\int q_1\,\mathrm{d}z\right)\right| \gg \left|\exp\left(\int q_j\,\mathrm{d}z\right)\right|, \qquad j = 2, 3, 4.$$

These levels of dominancy have been illustrated for particular equations by various diagrams (Heading 1957). The appropriate error analysis must be based on the corresponding integral equations.

(ii) If

$$\left|\exp\left(\int q_2\,\mathrm{d}z\right)\right| \gg \left|\exp\left(\int q_1\,\mathrm{d}z\right)\right| \gg \left|\exp\left(\int q_3\,\mathrm{d}z\right)\right|, \left|\exp\left(\int q_4\,\mathrm{d}z\right)\right|,$$

the same forms (12) persist satisfying the basic inequalities provided the arbitrary constant A_2 in f_2 is appropriately placed equal to zero. Solution (12) then presents the third level of dominancy.

(iii) If

$$\left|\exp\left(\int q_2\,\mathrm{d}z\right)\right|, \left|\exp\left(\int q_3\,\mathrm{d}z\right)\right| \gg \left|\exp\left(\int q_1\,\mathrm{d}z\right)\right| \gg \left|\exp\left(\int q_4\,\mathrm{d}z\right)\right|,$$

the same forms (12) persist if arbitrary constants A_2 and A_3 are placed equal to zero in f_2 and f_3 respectively, giving a solution presenting a second level of dominancy.

(iv) If

$$\left|\exp\left(\int q_j\right)\right| \gg \left|\exp\left(\int q_1\,\mathrm{d}z\right)\right|, \qquad j = 2, 3, 4,$$

forms (12) persist provided the constants A_2, A_3 and A_4 vanish; this is now a subdominant solution (or first level of dominancy). In all cases the basic inequalities between f_1, f_2, f_3, f_4 are satisfied. In all four cases, the analysis refers to the necessary conditions via the differential equations. If the inequalities are not satisfied, the solutions in the stated forms do not exist. Sufficiency analysis via the integral equations is similar to our previous investigation, but is more complicated to write out.

If, on the other hand, f_1 and f_2 satisfy neither $|f_1| \gg |f_2|$ nor $|f_1| \ll |f_2|$, consider a domain in which

$$|f_1|, |f_2| \gg |f_3|, |f_4|, \tag{13}$$

in which case we have the approximate equations

$$f_i' = q_i f_i + \Gamma_{i1} f_1 + \Gamma_{i2} f_2.$$

As before, introduce $\boldsymbol{f} = \exp(\int \Gamma_{11}\,\mathrm{d}z)\boldsymbol{g}$, giving, after the legitimate neglect of some coupling terms,

$$g_1' = q_1 g_1, \qquad g_2' = q_2 g_2 + \Gamma_{21} g_1 + (\Gamma_{22} - \Gamma_{11}) g_2,$$

$$g_3' = q_3 g_3 + \Gamma_{31} g_1 + \Gamma_{32} g_2, \qquad g_4' = q_4 g_4 + \Gamma_{41} g_1 + \Gamma_{42} g_2.$$

We obtain the following solutions:

$$f_1 = A_1 \exp\left(\int (q_1+\Gamma_{11})\,\mathrm{d}z\right), \qquad f_2 = A_2 \exp\left(\int (q_2+\Gamma_{22})\,\mathrm{d}z\right),$$

$$f_3 = f_1\mathrm{O}(1/k) \text{ or } f_2\mathrm{O}(1/k), \qquad f_4 = f_1\mathrm{O}(1/k) \text{ or } f_2\mathrm{O}(1/k).$$

Different cases arise depending on the relative dominancy of f_1 and f_2 compared with other exponential terms.

(i) When integration takes place along an anti-Stokes line for which

$$\left|\exp\left(\int q_1\,\mathrm{d}z\right)\right| = \left|\exp\left(\int q_2\,\mathrm{d}z\right)\right| \gg \left|\exp\left(\int q_3\,\mathrm{d}z\right)\right|, \left|\exp\left(\int q_4\,\mathrm{d}z\right)\right|,$$

inequalities (13) are consistently satisfied. Dominancy levels 3 and 4 are involved for the f_i.

(ii) If

$$\left|\exp\left(\int q_3\,\mathrm{d}z\right)\right| \gg \left|\exp\left(\int q_1\,\mathrm{d}z\right)\right| = \left|\exp\left(\int q_2\,\mathrm{d}z\right)\right| \gg \left|\exp\left(\int q_4\,\mathrm{d}z\right)\right|,$$

the same set of solutions is valid provided the arbitrary constant A_3 that would appear in g_3 is suitably placed equal to zero. Dominancy levels 2 and 3 are involved, with level 4 entirely cut out.

(iii) If

$$\left|\exp\left(\int q_3\,\mathrm{d}z\right)\right|, \left|\exp\left(\int q_4\,\mathrm{d}z\right)\right| \gg \left|\exp\left(\int q_1\,\mathrm{d}z\right)\right| = \left|\exp\left(\int q_2\,\mathrm{d}z\right)\right|,$$

the same set of solutions is valid provided the arbitrary constants A_3 and A_4 are suitably placed equal to zero. Dominancy levels 1 and 2 are entirely cut out, but 3 and 4 are involved on equal terms.

Our overall investigation has shown briefly from the differential equations: (i) necessary conditions when coupling terms can be neglected; (ii) that such neglect depends on the complete coupling terms and not merely on the coupling coefficients; (iii) why a change in the dependent variable should sometimes be made; (iv) why and how inequalities must be consistently satisfied between the f_i in a specified domain; (v) why not all the q_i appear in the four f_i for any particular solution; (vi) why some arbitrary constants must vanish, depending on the relative magnitudes of the exponentials of the integrals of the q_i; (vii) why no lower dominancy levels can appear in any solution; (viii) how the theory can be extended to equations of higher order; (ix) how contributions to the phase-memory terms must arise from the diagonal coupling coefficients which cannot be neglected. Any investigation of a specific problem must take all these points into consideration, backed by an analytical or numerical examination of the inequalities. Sufficiency analysis must be based on the integral equations, though this is often overlooked in physical problems.

References

Berry M V and Mount K E 1972 *Rep. Prog. Phys.* **35** 316–97

Budden K G 1975 *Phil Trans. R. Soc.* A **280** 111–30

Budden K G and Smith M S 1976 *Proc. R. Soc.* A **350** 27–46

Heading J 1957 *Proc. Camb. Phil. Soc.* **53** 419–41
—— 1962 *An Introduction to Phase-Integral Methods* (London: Methuen)
—— 1975 *Ordinary Differential Equations, Theory and Practice* (London: Elek Science)
—— 1977 *Q. J. Mech. Appl. Math.* **30** 281 -302
Jeffreys H 1962 *Asymptotic Approximations* (Oxford: UP)
Olver F W J 1978 *Phil. Trans. R. Soc.* A **289** 501–48
Smith M S 1975 *Proc. R. Soc.* A **346** 59–79